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Normal Liquid Helium-3 in the Static Fluctuation Approximation

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In this paper normal liquid helium-3 is studied for the first time within the framework of the so-called *static fluctuation approximation*. This is based on the replacement of the *square* of the local-field operator with its mean value. A closed set of nonlinear integral equations is derived for neutral many-fermionic systems. This set is solved numerically by an iteration method for a realistic interhelium potential. The thermodynamic properties are then obtained for normal liquid helium-3. The quadratic-fluctuation approximation is found to be valid for this system in the low-temperature limit (≤ 0.25 K). Our results are presented in a set of figures. The role of the interaction is emphasized, and the functional dependence on the temperature of key thermodynamic quantities is derived for normal liquid helium-3.

KEY WORDS: normal liquid helium-3; static fluctuation approximation.

1. INTRODUCTION

This paper is concerned with calculating from first principles the thermodynamic properties of *normal* liquid helium-3, which is the archetype strongly-interacting neutral many-fermionic system.

Since the early 1970s, it has been the obviously more exciting *superfluid*—rather than *normal*—liquid ^{3}He that has attracted attention more. However, normal liquid ^{3}He remains an interesting quantum fluid in its own right (Wilks, 1967).

Many microscopic techniques have been used over the years to study this system. These techniques can be classified into two broad types: The first type is perturbative (Bishop *et al.*, 1976; Burkhardt, 1968; Ghassib, 1974; Ghassib *et al.*, 1974, 1976; Mishra and Sreeram, 2000; Østgaard, 1968). The second type is variational, especially the correlated-basis-functions framework (Campbell *et al.*,

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1984; Feenberg, 1969). The aim of microscopic theory is, of course, to justify and explain phenomenological theory—Landau's (Pines and Noziéres, 1966).

Here we shall apply for the first time in this context the so-called static fluctuation approximation (SFA). The SFA has already been used to study various systems—including the classical two-dimensional Ising model (Nigmatullin *et al.*, 2000a); the proton model of ferroelectrics with tunneling (Nigmatullin *et al.*, 2000b); the one-, two-, and three-dimensional Ising model (Nigmatullin *et al.*, 2000c); and liquid helium-4 (Al-Sugheir *et al.*, 2001). Avoiding, as it does, Green's functions (Girish and Yia-Chung, 1998) and the highly-involved diagrammatic techniques, this approach is relatively simple compared to other conventional many-body approaches.

The underlying key physical idea is to substitute the square of the quadratic fluctuation operator with its mean value. The physical implication is that the true quantum-mechanical spectrum of this operator is replaced with a distribution around the expectation value of the local-field operator (Al-Sugheir *et al.*, 2001; Nigmatullin *et al.*, 2000a,b,c; Nigmatullin and Toboev, 1989).

We shall invoke this approximation to calculate the thermodynamic properties of normal liquid helium-3. To this end, it is necessary to obtain the so-called self-consistent *long-range equation.*

In this paper, we shall consider an extended uniform normal liquid helium-3 system of N ^{3}He atoms, each of mass m, occupying a volume Ω. This volume is large enough for the surface effects to be neglected, and each atom is characterized by its linear momentum $\vec{k}$ and spin $\lambda = \pm 1/2$. The translational invariance implies that the single-particle wavefunctions are plane waves (Fetter and Walecka, 1971). The chemical potential for liquid ^{3}He ($|\mu| = 2.52$ K/atom) is smaller than the electronic excitation energies. The minimum ionization energy for ^{3}He atom $\approx$24.58 eV, where l eV $= 1.1605 \times 10^4$ K. Hence, the excited states of the atoms need not appear explicitly in our theoretical description of our system; in this sense, these atoms are the elementary particles of our system. Superfluid ^{3}He is beyond the scope of this paper; to study critical phenomenon in this system we need orders of fluctuations higher than the quadratic order.

The synopsis of the rest of this paper is as follows. In Section 2 the closed set of nonlinear integral equations central to the present theoretical framework is derived for a neutral Fermi system. Section 3 is devoted to the calculations and numerics. Section 4 summarizes the results for normal liquid helium-3. Finally, in Section 5, the paper closes with some concluding remarks.

2. CLOSED SET OF NONLINEAR INTEGRAL EQUATIONS FOR A NEUTRAL FERMI SYSTEM

The total Hamiltonian describing a neutral many-fermionic system can be written as a linear combination of the local-field operator $\hat{E}_{k\lambda}$ and the number-of-particles operator (Al-Sugheir *et al.*, 2001):

$$\hat{H} = \sum_{\vec{k}\lambda} \hat{E}_{k\lambda}\hat{a}^{+}_{k\lambda}\hat{a}_{k\lambda}, \tag{1}$$

the indices $k\lambda$ denoting the complete set of quantum numbers describing a specific state.

The equation of motion of an operator $\hat{a}^{+}_{k\lambda}(\tau)$ in the Heisenberg representation can be written in the form

$$\frac{d\hat{a}^{+}_{k\lambda}}{d\tau} = (\hat{H}, \hat{a}^{+}_{k\lambda}), \tag{2}$$

where $\tau \equiv it$. According to the well-known anticommutation relations for a Fermi system

$$\left(\hat{a}_{k\lambda}, \hat{a}^{+}_{q\lambda_1}\right)_{+} = \delta_{kq}\delta_{\lambda\lambda_1}; \quad \left(\hat{a}_{k\lambda}, \hat{a}_{q\lambda_1}\right)_{+} = 0. \tag{3}$$

Assuming that the local-field operator $\hat{E}_{k\lambda}$ is hermitian and that it commutes with creation and annihilation operators, we can write

$$\frac{d\hat{a}^{+}_{k\lambda}}{d\tau} = (\hat{H}, \hat{a}^{+}_{k\lambda})_{-} = \hat{E}_{k\lambda}\hat{a}^{+}_{k\lambda}. \tag{4}$$

The total Hamiltonian describing the neutral many-fermionic system can be written as

$$\begin{aligned}\hat{H} &= \int d\vec{r}\,\hat{\Psi}^{+}(\vec{r})\left(-\frac{\hbar^2}{2m}\nabla^2\right)\hat{\Psi}(\vec{r}) \\ &\quad + \frac{1}{2}\int\!\!\int d\vec{r}_1\,d\vec{r}_2\hat{\Psi}^{+}(\vec{r}_1)\hat{\Psi}^{+}(\vec{r}_2)V(\vec{r}_1 - \vec{r}_2)\hat{\Psi}(\vec{r}_2)\hat{\Psi}(\vec{r}_1). \end{aligned} \tag{5}$$

Here $\hat{\Psi}(\vec{r})$ and $\hat{\Psi}^{+}(\vec{r})$ are the field operators, h is Planck's constant ($\hbar \equiv h/2\pi \equiv$ Dirac's constant), m is the femionic mass, and $V(\vec{r}_1 - \vec{r}_2)$ is the pairwise central potential that depends only on the modulus of $|\vec{r}_1 - \vec{r}_2|$.

It is convenient to write the field operators as linear combinations of the creation and annihilation operators:

$$\hat{\Psi}(\vec{r}) = \sum_{\vec{k}\lambda} \psi_{\vec{k}\lambda}(\vec{r})\hat{a}_{\vec{k}\lambda}; \tag{6}$$

$$\hat{\Psi}^{+}(\vec{r}) = \sum_{\vec{k}\lambda} \psi^{+}_{\vec{k}\lambda}(\vec{r})\hat{a}^{+}_{\vec{k}\lambda}, \tag{7}$$

where the coefficients $\psi_{\vec{k}\lambda}(\vec{r})$, $\psi^{+}_{\vec{k}\lambda}(\vec{r})$ are the single-particle wavefunctions and the sum is over the complete set of single-particle quantum numbers. In particular, the index $\vec{k}$ denotes the linear momentum of the particle and the index λ represents the spin of the particle.

In a uniform infinite system all physical properties must, of course, be invariant under spatial translations. This suggests that the single-particle wavefunctions are plane waves:

$$\psi_{\vec{k}\lambda}(\vec{r}) = \frac{1}{\sqrt{\Omega}} \exp(i\vec{k} \cdot \vec{r})\eta_\lambda, \tag{8}$$

where Ω is the normalization volume of the system and η_λ are the two spin wavefunctions.

Integration over the spatial coordinates of (5) gives the Hamiltonian in the second-quantization representation:

$$\hat{H} = \sum_{\vec{k}\lambda} \frac{\hbar^2 k^2}{2m} \hat{a}^+_{k\lambda}\hat{a}_{k\lambda} + \frac{1}{2\Omega} \sum_{\vec{k}\vec{q}\vec{p},\lambda\lambda_1} V(k)\hat{a}^+_{p-k\lambda}\hat{a}_{p\lambda}\hat{a}^+_{q\lambda_1}\hat{a}_{q-k\lambda_1}, \tag{9}$$

$V(k)$ being the Fourier transform of the pair potential defined as

$$V(k) = \int V(r) \exp(i\vec{k} \cdot \vec{r})\, d\vec{r}. \tag{10}$$

The grand canonical Hamiltonian of this system is

$$\hat{H} = \sum_{\vec{k}\lambda} \varepsilon(k)\hat{a}^+_{k\lambda}\hat{a}_{k\lambda} + \frac{1}{2\Omega} \sum_{\vec{k}\vec{q}\vec{p},\lambda\lambda_1} V(k)\hat{a}^+_{p-k\lambda}\hat{a}_{p\lambda}\hat{a}^+_{q\lambda_1}\hat{a}_{q-k\lambda_1}, \tag{11}$$

where $\varepsilon(k) \equiv (\hbar^2 k^2/2m) - \mu$, μ being the chemical potential.

From (4) and (9) we find that

$$\hat{E}_{k\lambda} = [\hat{a}_{k\lambda}, (\hat{H}, \hat{a}^+_{k\lambda})_-]_+ = \varepsilon(k) + \frac{1}{\Omega} \sum_{\vec{q}\lambda_1} \left[V(0) - V(\vec{q} - \vec{k})\delta_{\lambda_1\lambda}\right]\hat{a}^+_{q\lambda_1}\hat{a}_{q\lambda_1}. \tag{12}$$

From this equation we find that the local-field operator is spin independent: $\hat{E}_{k\lambda_1} = \hat{E}_{k\lambda_2} = \hat{E}_k$. This is reasonable since the interaction potential $V(\vec{r}_1 - \vec{r}_2)$ is spin independent, and so is the particles distribution operator: $\hat{n}_{q\lambda_1} = \hat{a}^+_{q\lambda_1}\hat{a}_{q\lambda_1} = \hat{a}^+_{q\lambda_2}\hat{a}_{q\lambda_2}$.

The local-field operator in the ground state is

$$\hat{E}_0 = \frac{1}{\Omega} \sum_{\vec{q}\lambda_1} \left[V(0) - V(\vec{q})\delta_{\lambda\lambda_1}\right]\hat{a}^+_{q\lambda_1}\hat{a}_{q\lambda_1}. \tag{13}$$

The excitation local-field operator, as measured relative to $\hat{E}_0$, is

$$\hat{\mathcal{E}}_{\vec{k}} \equiv \hat{E}_{\vec{k}} - \hat{E}_0 = \varepsilon(k) + \frac{1}{\Omega}\left[\sum_{\vec{q}} W(\vec{k}, \vec{q})\hat{n}_q\right], \tag{14}$$

and the corresponding mean value is

$$\langle \hat{E}_{\vec{k}} \rangle = \varepsilon(k) + \frac{1}{\Omega} \left[\sum_{\vec{q}} W(\vec{k}, \vec{q}) \langle \hat{n}_q \rangle \right], \tag{15}$$

where

$$W(\vec{k}, \vec{q}) \equiv V(\vec{q}) - V(\vec{k} - \vec{q}), \quad \text{and} \quad \hat{n}_{q\lambda} \equiv \hat{n}_q = \hat{a}^{+}_{q\lambda} \hat{a}_{q\lambda}.$$

In the SFA the square of the quadratic fluctuation operator is replaced with its mean value:

$$(\Delta \hat{E}_k)^2 \cong \langle (\Delta \hat{E}_k)^2 \rangle. \tag{16}$$

$\Delta \hat{E}_k \equiv \hat{E}_k - \langle \hat{E}_k \rangle$ is the corresponding deviation of the mean-field operator from its mean value. It follows that the local-field deviation operator has two symmetric eigenvalues:

$$(\Delta \hat{E}_k)^2 \cong \langle (\Delta \hat{E}_k)^2 \rangle = \varphi_k^2, \tag{17}$$

which, in fact, defines a splitting of the operator $\Delta \hat{E}_k$ produced by the quadratic fluctuations.

To obtain the desired closed system of nonlinear integral equations, it is necessary to find the creation operator in terms of the local-field operator. The general solution of (4) is

$$\hat{a}^{+}_{k\lambda}(\tau) = \hat{a}^{+}_{k\lambda} \exp(\hat{E}_k \tau) = \hat{a}^{+}_{k\lambda} \exp[(\langle \hat{E}_k \rangle + \Delta \hat{E}_k)\tau]. \tag{18}$$

It is more convenient to rewrite (18) as linear in terms of the fluctuations of the local-field operator; this could be done with the aid of the identity:

$$B(a + b\Delta \hat{E}_k) \equiv \eta_0(k) + \eta_1(k) \Delta \hat{E}_k, \tag{19}$$

where

$$\eta_0(k) \equiv \frac{1}{2}[B(a + b\varphi_k) + B(a - b\varphi_k)]; \tag{20a}$$

$$\eta_1(k) \equiv \frac{1}{2\varphi_k}[B(a + b\varphi_k) - B(a - b\varphi_k)]. \tag{20b}$$

According to this identity we can write (18) in the final form:

$$\hat{a}^{+}_{k\lambda}(\tau) = \hat{a}^{+}_{k\lambda} \exp(\langle \hat{E}_k \rangle \tau) \left[\cosh(\varphi_k \tau) + \frac{\Delta \hat{E}_k}{\varphi_k} \sinh(\varphi_k \tau) \right]. \tag{21}$$

From this solution we can obtain the so-called *long-range equation*. To this end, we use the identity:

$$\langle \hat{C}(\beta) \hat{B} \rangle = \langle \hat{B} \hat{C} \rangle = \frac{1}{Q} \mathrm{Tr}[\exp(-\beta \hat{H}) \hat{B} \hat{C}], \tag{22}$$

where Q is the grand partition function of the system. Here and below we shall replace τ with $\beta = (1/k_B T)$, k_B being Boltzmann's constant and T the absolute temperature; this is common in statistical quantum mechanics (Feynman, 1972; Kadanoff and Baym, 1962).

Assuming that $\hat{C}(\beta) = \hat{a}^+_{k\lambda}(\beta)$ and $\hat{B} = \hat{a}_{k\lambda}\hat{A}$, we obtain

$$\langle \hat{a}^+_{k\lambda}(\beta)\hat{a}_{k\lambda}\hat{A}\rangle = \langle \hat{a}_{k\lambda}\hat{A}\hat{a}^+_{k\lambda}\rangle. \tag{23}$$

If the operator $\hat{A}$ is chosen to commute with $\hat{a}^+_{k\lambda}$, $\hat{a}_{k\lambda}$, and $\hat{E}_k$, (23) can then be written as

$$\langle \hat{a}^+_{k\lambda}(\beta)\hat{a}_{k\lambda}\hat{A}\rangle = \langle \hat{a}_{k\lambda}\hat{A}\hat{a}^+_{k\lambda}\rangle = \langle \hat{a}_{k\lambda}\hat{a}^+_{k\lambda}\hat{A}\rangle. \tag{24}$$

From this equation and (21) we obtain

$$\begin{aligned}\langle \hat{A}\rangle &= \langle \hat{n}_k\hat{A}\rangle[\exp(\beta\langle \hat{E}_k\rangle)\ \cosh(\beta\varphi_k) + 1] \\ &\quad + \langle \hat{n}_k\Delta\hat{E}_k\hat{A}\rangle\ \exp(\beta\langle \hat{E}_k\rangle)\ \frac{\sinh(\beta\varphi_k)}{\varphi_k}.\end{aligned} \tag{25}$$

If, in (25), $\hat{A} \to \Delta\hat{E}_k\hat{A}$, we obtain the additional equation:

$$\begin{aligned}\langle \Delta\hat{E}_k\hat{A}\rangle &= \langle \hat{n}_k\Delta\hat{E}_k\hat{A}\rangle[\exp(\beta\langle \hat{E}_k\rangle)\ \cosh(\beta\varphi_k) + 1] \\ &\quad + \varphi_k\ \sinh(\beta\varphi_k)\ \exp(\beta\langle \hat{E}_k\rangle)\langle \hat{n}_k\hat{A}\rangle.\end{aligned} \tag{26}$$

To obtain the long-range equation, $\langle \hat{n}_k\Delta\hat{E}_k\hat{A}\rangle$ is eliminated from (25) and (26); so that

$$\langle \hat{n}_k\hat{A}\rangle = \eta_0(k)\langle \hat{A}\rangle + \eta_1(k)\langle \Delta\hat{E}_k\hat{A}\rangle, \tag{27}$$

where

$$\eta_0(k) \equiv \frac{1}{2}\left\{\frac{1}{\exp[\beta(\langle \hat{E}_k\rangle + \varphi_k)] + 1} + \frac{1}{\exp[\beta(\langle \hat{E}_k\rangle - \varphi_k)] + 1}\right\}; \tag{28a}$$

$$\eta_1(k) \equiv \frac{1}{2\varphi_k}\left\{\frac{1}{\exp[\beta(\langle \hat{E}_k\rangle + \varphi_k)] + 1} - \frac{1}{\exp[\beta(\langle \hat{E}_k\rangle - \varphi_k)] + 1}\right\}. \tag{28b}$$

We can now find the closed system of nonlinear integral equations from the long-range equation (27). First, putting $\hat{A} = 1$ in (27), we have the particle distribution, where the quadratic fluctuations are symmetric, $\langle \Delta\hat{E}_k\rangle = 0$:

$$\langle \hat{n}_k\rangle = \eta_0(k). \tag{29}$$

It is more convenient to rewrite the long-range equation (27) in terms of the fluctuations of the occupation-number operator, this being defined as

$$\hat{n}_k \equiv \langle \hat{n}_k\rangle - \Delta\hat{n}_k, \tag{30}$$

where we are have taken the local-field operator as $\hat{E}_k = \langle \hat{E}_k \rangle + \Delta \hat{E}_k$. Therefore, the occupation-number operator must be defined as in (30); this is because there is a relative sign between the fluctuations in the number-of-particles operator and the fluctuations in the local-field operator. So if the fluctuations in the local-field operator lead to an increase in the state energy, then the fluctuations in the number-of-particles operator lead to a decrease in the number of particles in this state. In terms of the fluctuations of the occupation-number operator, the long-range equation is

$$\langle \Delta \hat{n}_k \hat{A} \rangle = -\eta_1(k) \langle \Delta \hat{E}_k \hat{A} \rangle. \tag{31}$$

Putting $\hat{A} = \Delta \hat{n}_q$ in this equation, where $q \neq k$, we have the pair correlation function $\langle \Delta \hat{n}_k \Delta \hat{n}_q \rangle_c$, the index c denoting the true correlations $q \neq k$:

$$\begin{aligned}\langle \Delta \hat{n}_k \Delta \hat{n}_q \rangle_c &= -\eta_1(k) \langle \Delta \hat{E}_k \Delta \hat{n}_q \rangle_c \\ &= -\frac{\eta_1(k)}{\Omega} \sum_{\vec{p}} W(\vec{k}, \vec{p}) \langle \Delta \hat{n}_p \Delta \hat{n}_q \rangle. \end{aligned}\tag{32}$$

The correlation function $\langle \Delta \hat{n}_p \Delta \hat{n}_q \rangle$ can be written as

$$\langle \Delta \hat{n}_p \Delta \hat{n}_q \rangle = \langle (\Delta \hat{n}_q)^2 \rangle \delta_{qp} + \langle \Delta \hat{n}_p \Delta \hat{n}_q \rangle_c. \tag{33}$$

The value $\langle (\Delta \hat{n}_q)^2 \rangle$ is determined as

$$\langle (\Delta \hat{n}_q)^2 \rangle = \left\langle \hat{n}_q^2 \right\rangle - \langle \hat{n}_q \rangle^2. \tag{34}$$

For fermionic systems, $\hat{n}_q^2 = \hat{n}_q$; so that (34) becomes

$$\langle (\Delta \hat{n}_q)^2 \rangle = \langle \hat{n}_q \rangle (1 - \langle \hat{n}_q \rangle). \tag{35}$$

To close our system of nonlinear integral equations, putting $\hat{A} = \Delta \hat{E}_k$ in (31), we have

$$-\eta_1(k) \varphi_k^2 = \frac{1}{\Omega} \sum_{\vec{p}} W(\vec{k}, \vec{p}) \langle \Delta \hat{n}_p \Delta \hat{n}_k \rangle. \tag{36}$$

The set of nonlinear integral equations (15), (29), (32), (35), and (36) can be solved numerically to calculate the thermodynamic properties of the system.

3. CALCULATIONS

In the thermodynamic limit the summation in (15), (32), and (36) can be changed to an integration. After integrating over the solid angle, these equations reduce to

$$\langle \hat{E}_k \rangle = \varepsilon(k) + \frac{1}{2\pi^2} \int_0^\infty W(k, p) \langle \hat{n}_p \rangle p^2 \, dp; \tag{37}$$

$$\langle \Delta \hat{n}_k \Delta \hat{n}_q \rangle_c = -\frac{\eta_1(k)}{2\pi^2} \int_0^\infty W(k, p) \langle \Delta \hat{n}_p \Delta \hat{n}_q \rangle p^2 \, dp; \tag{38}$$

$$-\eta_1(k) \varphi_k^2 = \frac{1}{2\pi^2} \int_0^\infty W(k, p) \langle \Delta \hat{n}_p \Delta \hat{n}_k \rangle_c p^2 \, dp, \tag{39}$$

where $W(k, p) \equiv V(p, 0) - V(k, p)$, $V(k, p)$ being the Fourier-Bessel transform of the potential, defined as

$$V(k, p) = 4\pi \int_0^\infty V(r) \frac{\sin(kr)}{kr} \frac{\sin(pr)}{pr} r^2 \, dr. \tag{40}$$

The integrand in (37)–(39) is calculated by Gaussian quadrature (Ali, 1997; Bishop *et al.*, 1977; Ghassib *et al.*, 1976). Our set of nonlinear integral equations will be solved numerically by an iteration method for a realistic interhelium potential: FDD-1 potential of Frost and Musulin (Bishop *et al.*, 1977; Bruch and McGee, 1967, 1970) because it retains most of the desirable features of HFDHE2 (Aziz *et al.*, 1979; Janzen and Aziz, 1995), generally regarded as the best He–He potential available, but is simpler to handle. The Fourier-Bessel transformation of this potential, $V(k, q)$, is given by Ali (1997). Throughout our calculations a natural system of units is used, such that $\hbar = 1 = m$, where m is the ^{3}He atomic mass, the conversion factor being $\hbar^2/m = 16.0838$ KÅ^{-1} (Bishop *et al.*, 1977). The chemical potential of liquid helium-3, which plays a crucial role in calculating the number-of-particles distribution, must be known as a function of temperature. To this end we use

$$\mu = \left(\frac{\partial E}{\partial N} \right)_S, \tag{41}$$

E being the internal energy of the system and S the entropy of the system. Using elementary thermodynamics, we may write this equation in the form

$$\mu = \left(\frac{\partial E}{\partial N} \right)_T - T \left(\frac{\partial S}{\partial N} \right)_T. \tag{42}$$

At low temperatures, the second term of (42) is negligible compared to the first (Pines and Noziéres, 1966).

To calculate the thermodynamic properties of the system, we should calculate the grand partition function Q. The usual definition is

$$\begin{aligned} Q &= \mathrm{Tr}[\exp(-\beta \hat{H})] \\ &= \sum_{\hat{n}_p} \exp\left(-\beta \sum_{\vec{p}\lambda} \hat{E}_p \hat{n}_p \right) \\ &= \prod_{\vec{p}\lambda} \sum_{\hat{n}_p} \exp(-\beta \hat{E}_p \hat{n}_p). \end{aligned} \tag{43}$$

For Fermi systems $n_p = 0$ or 1; so that (43) becomes

$$Q = \prod_{p\lambda} [1 + \exp(-\beta \hat{E}_p)]. \tag{44}$$

It is more convenient to take the logarithms of both sides:

$$\ln Q = \sum_{\vec{p}\lambda} \ln[1 + \exp(-\beta \hat{E}_p)]. \tag{45}$$

Using the identity (19), we have

$$\ln Q = \sum_{\vec{p}\lambda} [q_0(p) + q_1(p)\Delta \hat{E}_p]. \tag{46}$$

Taking into account the symmetry of the quadratic fluctuations of the local-field operator, we get from (46)

$$\ln Q = \sum_{\vec{p}\lambda} q_0(p),$$

where

$$q_0(p) = \frac{1}{2} \begin{bmatrix} \ln\{1 + \exp[-\beta(\langle \hat{E}_p \rangle + \varphi_p)]\} \\ + \ln\{1 + \exp[-\beta(\langle \hat{E}_p \rangle - \varphi_p)]\} \end{bmatrix}. \tag{47}$$

The grand mean energy $\langle \hat{H} \rangle$ is defined as

$$\langle \hat{H} \rangle = -\frac{\partial \ln Q}{\partial \beta} = \sum_{\vec{p}\lambda} \left[\langle \hat{E}_p \rangle \langle \hat{n}_p \rangle + \eta_1(p)\varphi_p^2 \right]. \tag{48}$$

From the grand partition function and the grand mean energy it is simple to evaluate the other thermodynamic properties.

The usual definition of the pressure is given by (Huang, 1987; Pathria, 1992)

$$P = k_B T \frac{\ln Q}{\Omega}. \tag{49}$$

The entropy of the system can be evaluated from the first law of thermodynamics in terms of the grand mean internal energy U and the pressure:

$$S = \frac{1}{T}(U + P\Omega). \tag{50}$$

Finally, the specific heat capacity of the system at constant volume is

$$C_v = \left(\frac{\partial U}{\partial T} \right)_\Omega. \tag{51}$$

4. RESULTS AND DISCUSSION

From numerical calculations we find that our closed system of nonlinear integral equations is divergent at temperatures $T \geq 0.25$ K; the fluctuations in the local-field operator increase with temperature. This means that the SFA is valid for strongly-interacting Fermi systems only at temperatures lower than some temperature ($\approx$0.25 K for normal liquid helium-3). At higher temperatures and for more accurate calculations of the long-range equation, higher orders of the fluctuations must be considered. These orders are presumably also necessary to handle the superfluid transitions that the system undergoes for $T \leq 2.7$ mK. It seems that this conclusion is generally true for critical phenomena within the *quadratic* fluctuation approximation (Nigmatullin *et al.*, 2000c).

The strength of the interaction potential of the system also plays a crucial role in determining the limits of validity of the *quadratic* fluctuation approximation (Al-Sugheir *et al.*, 2001). For dilute systems the validity of the SFA extends to higher

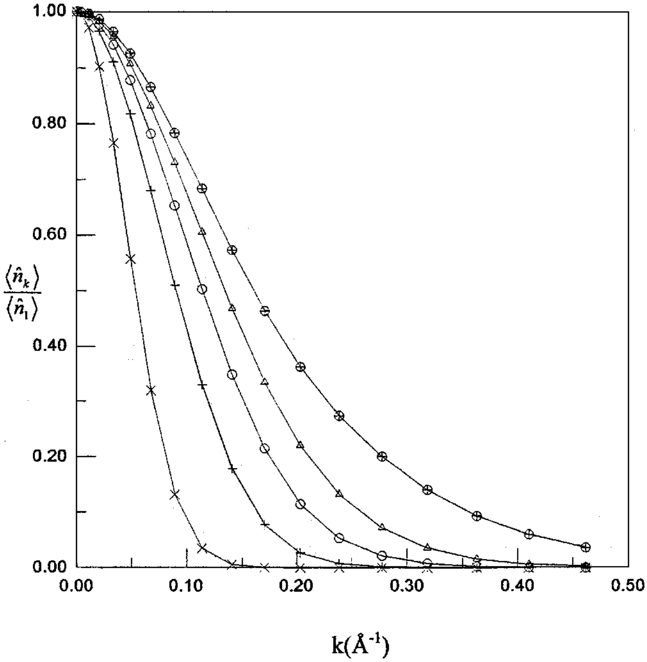

Fig. 1. Distribution of the number of particles in the different states relative to the lowest state, $\langle \hat{n}_k \rangle / \langle \hat{n}_1 \rangle$, for the FDD-1 potential at temperatures $T_1 = 0.03$ K ($\times$), $T_2 = 0.09$ K ($+$), $T_3 = 0.15$ K ($\circ$), $T_4 = 0.21$ K ($\triangle$), and $T_5 = 0.25$ K ($\oplus$), versus the relative momentum (k).

temperatures. The central He–He potential can be expressed in the form

$$V(r) = Af(r), \tag{52}$$

where A is a factor determining the strength (amplitude) of the potential and $f(r)$ is a "shape function" describing the overall shape of the potential (Ghassib, 1984). When the strength factor equals 1 ($A = 1$), the shape function $f(r)$ becomes the realistic interhelium potential. In a similar manner to liquid ^{4}He (Al-Sugheir *et al.*, 2001), the fluctuations in the local-field operator at fixed temperature increase with A. As A increases, more states become occupied; so that the fluctuations in the local-field operator of state $|k\rangle$ increase. Also, the fluctuations in the local-field operator at fixed strength factor increase with temperature. However, if the fluctuations in the local-field operator of state $|k\rangle$ exceed a certain value compared to the mean value of the local-field operator of the same state, the closed set of nonlinear integral equations become divergent.

Figure 1 shows the distribution of the number of fermions at different temperatures. As expected, the width of the distribution increases with temperature, since more states become occupied. This is explained by the increase in the

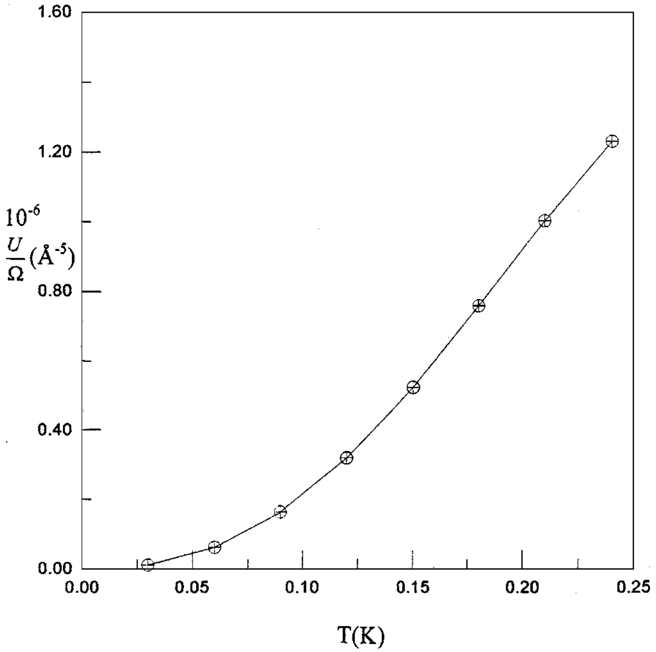

Fig. 2. The internal grand mean energy per unit volume U/Ω for normal liquid helium-3 as a function of temperature (T).

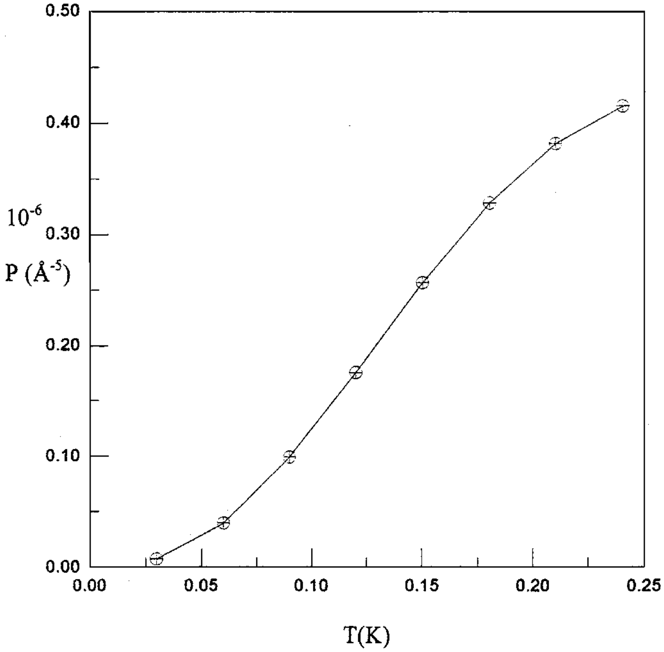

Fig. 3. The pressure (P) for normal liquid helium-3 as a function of temperature (T).

fluctuations in the local-field operator as the temperature increases. The fluctuations in the local-field operator of state $|k\rangle$ depend on the interaction potential operating between this state and other occupied states through the correlation function $\langle \Delta n_k \Delta n_q \rangle_c$.

Figures 2–5 show the numerical calculations of the thermodynamic properties of normal liquid helium-3. Figure 2 shows the internal grand mean energy per unit volume up to 0.25 K. Fitting these data, we find that $U \propto T^{2.3}$. Thus, the interaction term leads to an enhanced temperature dependence of the internal grand mean energy; for ideal systems, $U \propto T^2$. Figure 3 shows the internal pressure of the system. The temperature dependence is $P \propto T^{1.5}$; whereas for ideal systems, $P \propto T^2$. For interacting systems, the particles become more localized compared to the ideal system; so that the internal pressure becomes less sensitive to the temperature. Figure 4 shows that the specific heat capacity is linear in the temperature up to $T = 0.15$ K; thereafter a slow fall occurs as the temperature increases. This result is consistent with other calculations based on the spin fluctuation theory (Mishra and Sreeram, 2000). Both theoretical (Misawa, 1999) and experimental

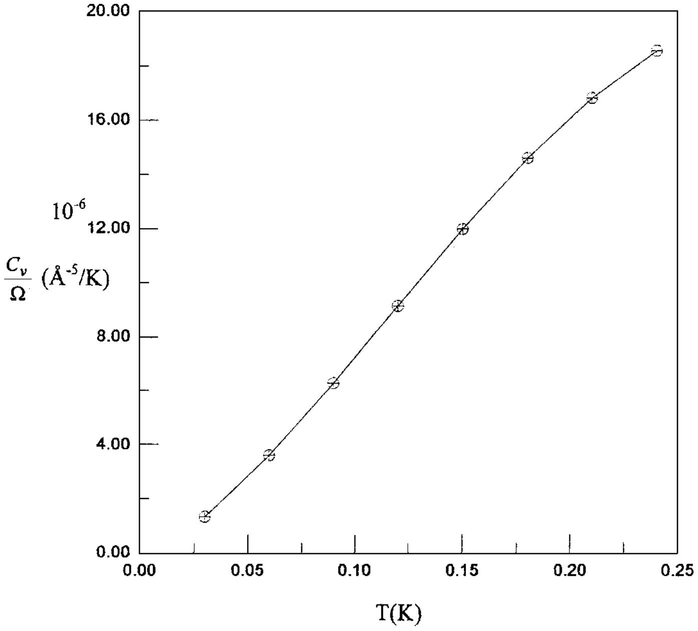

Fig. 4. The specific heat capacity per unit volume C_v/Ω for normal liquid helium-3 as a function of temperature (T).

(Masashi and Setsuo, 2000) studies of ^{3}He fluid films have shown that the heat capacity of these systems vary as $C(T) = \gamma_0 T - \gamma_2 T^2$ at low temperatures, where γ_0 and γ_2 are constants. Figure 5 shows the entropy of normal liquid helium-3, which has almost the same temperature dependence as the specific heat capacity.

5. CONCLUSION

The SFA has been used to study normal liquid ^{3}He—as an archetype of neutral strongly-interacting many-fermionic systems—for the first time. It has been found that the SFA is valid for this system only at temperatures $T \leq 0.25$ K. At the same time, however, no critical phenomena—including the superfluid transitions for $T \leq 2.7$ mK—could be predicted within the present *quadratic* fluctuation approximation. To predict such critical phenomena higher orders of fluctuations must be considered. The basic achievements of this paper are, then (1) the full derivation of the SFA, for the first time, for neutral many-fermionic systems, and (2) the calculation of the thermodynamic properties of the strongly-interacting normal liquid helium-3 at temperatures $\leq$0.25 K.

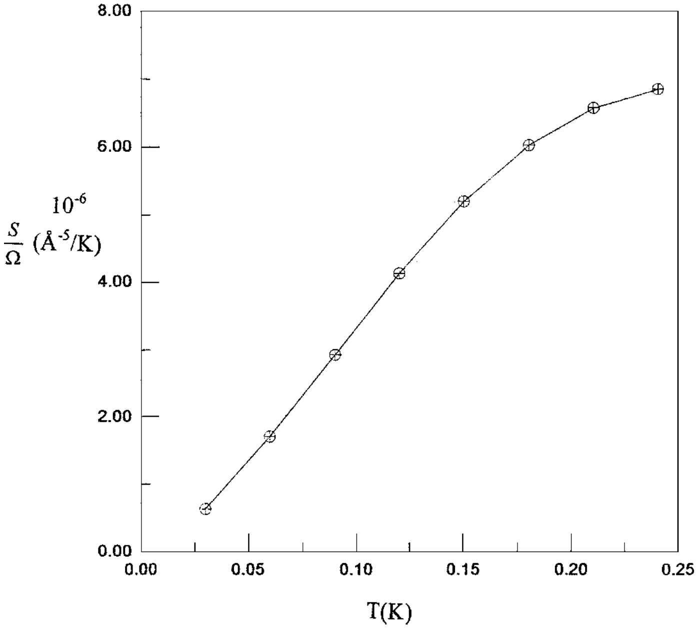

Fig. 5. The entropy per unit volume S/Ω for normal liquid helium-3 as a function of temperature (T).

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